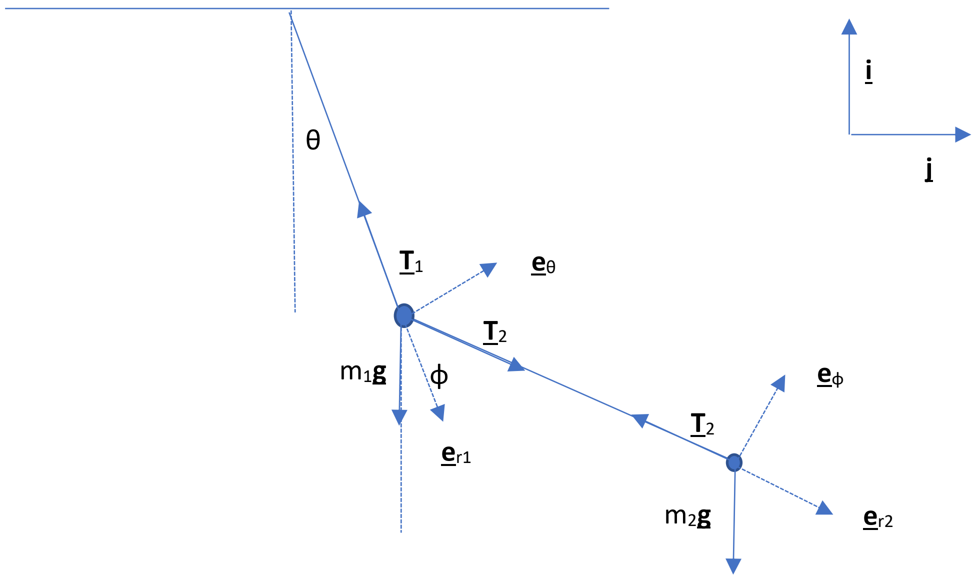
The Double Pendulum

Derivation of Suitable Relevant Equations

To simplify the analysis a little, I assume that both pendulum rods are the same length, a, but with different masses.



Firstly, some geometric relationships;

$\underline{\mathbf{e}}_\theta = \cos\theta\ \underline{\mathbf{i}} + \sin\theta\ \underline{\mathbf{j}}$

$\underline{\mathbf{e}}_{r1} = \sin\theta\ \underline{\mathbf{i}} - \cos\theta\ \underline{\mathbf{j}}$

$\underline{\mathbf{e}}_\phi = \cos\phi\ \underline{\mathbf{i}} + \sin\phi\ \underline{\mathbf{j}}$

$\underline{\mathbf{e}}_{r2} = \sin\phi\ \underline{\mathbf{i}} - \cos\phi\ \underline{\mathbf{j}}$

and, conversely $\quad \underline{\mathbf{j}} = -\cos\theta\ \underline{\mathbf{e}}_{r1} + \sin\theta\ \underline{\mathbf{e}}_\theta$

Also, the position vectors of the masses m_1 and m_2 at any time are given simply by;

$$\underline{\mathbf{r}}_1 = a\ \underline{\mathbf{e}}_{r1} \quad \text{and} \quad \underline{\mathbf{r}}_2 = a\ \underline{\mathbf{e}}_{r1} + a\ \underline{\mathbf{e}}_{r2}$$

The two equations of motion from Newton's Laws are;

$$m_1\ddot{\underline{\mathbf{r}}}_1 = -m_1 g\ \underline{\mathbf{j}} + T_2\ \underline{\mathbf{e}}_{r2} - T_1\ \underline{\mathbf{e}}_{r1} \quad \text{and} \quad m_2\ddot{\underline{\mathbf{r}}}_2 = -m_2 g\ \underline{\mathbf{j}} - T_2\ \underline{\mathbf{e}}_{r2}$$

The time derivatives of the position vectors can be expressed as follows;

$$\dot{\underline{\mathbf{r}}}_1 = a\ \dot{\underline{\mathbf{e}}}_{r1} = a\dot\theta(\cos\theta\ \underline{\mathbf{i}} + \sin\theta\ \underline{\mathbf{j}}) = a\dot\theta\underline{\mathbf{e}}_\theta$$

$$\ddot{\underline{\mathbf{r}}}_1 = a\ddot\theta\underline{\mathbf{e}}_\theta + a\dot\theta\dot{\underline{\mathbf{e}}}_\theta = a\ddot\theta\underline{\mathbf{e}}_\theta + a\dot\theta^2(-\sin\theta\ \underline{\mathbf{i}} + \cos\theta\ \underline{\mathbf{j}}) = -a\dot\theta^2\underline{\mathbf{e}}_{r1} + a\ddot\theta\underline{\mathbf{e}}_\theta$$

And,

$$\underline{\dot{\mathbf{r}}}_2 = a\,\underline{\dot{\mathbf{e}}}_{r1} + a\,\underline{\dot{\mathbf{e}}}_{r2} \;=\; a\dot{\theta}\underline{\mathbf{e}}_\theta + a\dot{\phi}(\cos\phi\,\underline{\mathbf{i}} + \sin\phi\,\underline{\mathbf{j}}) \;=\; a\dot{\theta}\underline{\mathbf{e}}_\theta + a\dot{\phi}\underline{\mathbf{e}}_\phi$$

$$\underline{\ddot{\mathbf{r}}}_2 = a\ddot{\theta}\underline{\mathbf{e}}_\theta + a\ddot{\phi}\underline{\mathbf{e}}_\phi + a\dot{\theta}\underline{\dot{\mathbf{e}}}_\theta + a\dot{\phi}\underline{\dot{\mathbf{e}}}_\phi \;=\; a\ddot{\theta}\underline{\mathbf{e}}_\theta + a\ddot{\phi}\underline{\mathbf{e}}_\phi - a\dot{\theta}^2\underline{\mathbf{e}}_{r1} - a\dot{\phi}^2\underline{\mathbf{e}}_{r2}$$

Now to substitute the expressions for the upward unit vector $\underline{\mathbf{j}}$ and the position vector derivatives into the equations of motion;

$$m_1(-a\dot{\theta}^2\underline{\mathbf{e}}_{r1} + a\ddot{\theta}\underline{\mathbf{e}}_\theta) \;=\; m_1 g(-\cos\theta\,\underline{\mathbf{e}}_{r1} + \sin\theta\,\underline{\mathbf{e}}_\theta) + T_2\,\underline{\mathbf{e}}_{r2} - T_1\,\underline{\mathbf{e}}_{r1}$$

And taking the scalar product of this equation with the unit vector $\underline{\mathbf{e}}_\theta$ leads to;

$$m_1 a\ddot{\theta} = -\,m_1 g\sin\theta + T_2\,\underline{\mathbf{e}}_{r2}.\underline{\mathbf{e}}_\theta$$

Now, $\underline{\mathbf{e}}_{r2}.\underline{\mathbf{e}}_\theta \;=\; (\sin\phi\,\underline{\mathbf{i}} - \cos\phi\,\underline{\mathbf{j}}).(\cos\theta\,\underline{\mathbf{i}} + \sin\theta\,\underline{\mathbf{j}}) \;=\; \sin\phi\cos\theta - \cos\phi\sin\theta \;=\; \sin(\phi - \theta)$

Thus, $$m_1 a\ddot{\theta} = -\,m_1 g\sin\theta + T_2\sin(\phi - \theta) \qquad **$$

Secondly, doing likewise for the other equation of motion gives;

$$m_2(a\ddot{\theta}\underline{\mathbf{e}}_\theta + a\ddot{\phi}\underline{\mathbf{e}}_\phi - a\dot{\theta}^2\underline{\mathbf{e}}_{r1} - a\dot{\phi}^2\underline{\mathbf{e}}_{r2}) \;=\; -\,m_2 g(-\cos\theta\,\underline{\mathbf{e}}_{r1} + \sin\theta\,\underline{\mathbf{e}}_\theta) - T_2\underline{\mathbf{e}}_{r2} \qquad ***$$

And taking the scalar product of this latest equation with the unit vector $\underline{\mathbf{e}}_\theta$ leads to;

$$m_2 a(\ddot{\theta} + \ddot{\phi}\underline{\mathbf{e}}_\phi.\underline{\mathbf{e}}_\theta - \dot{\phi}^2\sin(\phi - \theta)) \;=\; -\,m_2 g\,\underline{\mathbf{j}}.\underline{\mathbf{e}}_\theta - T_2\,\underline{\mathbf{e}}_{r2}.\underline{\mathbf{e}}_\theta$$

Working out the results of the unit vector scalar products from the geometric relationships then simplifies down to;

$$m_2 a(\ddot{\theta} + \ddot{\phi}\cos(\phi - \theta) - \dot{\phi}^2\sin(\phi - \theta)) \;=\; -\,m_2 g\sin\theta - T_2\sin(\phi - \theta)$$

Now, from equation ** above, $\quad T_2\sin(\phi - \theta) \;=\; m_1 g\sin\theta - m_1 a\ddot{\theta}$

Substituting this expression into the previous equation eliminates T_2 to give, after rearranging;

$$(m_1 + m_2)a\ddot{\theta} + m_2 a(\ddot{\phi}\cos(\phi - \theta) - \dot{\phi}^2\sin(\phi - \theta)) + (m_1 + m_2)g\sin\theta \;=\; 0$$

On setting $m = m_2/(m_1 + m_2)$ finally leads to;

$$\ddot{\theta} + m\ddot{\phi}\cos(\phi - \theta) - m\dot{\phi}^2\sin(\phi - \theta) + (g/a)\sin\theta \;=\; 0$$

or

$$\boldsymbol{\ddot{\theta} = -\,m\ddot{\phi}\cos(\phi - \theta) + m\dot{\phi}^2\sin(\phi - \theta) - (g/a)\sin\theta} \qquad \textbf{A1}$$

To get a second suitable equation, reconsider equation *** above – but this time take the scalar product with the unit vector $\underline{\mathbf{e}}_\phi$ so that;

$$m_2(a\ddot{\theta}\underline{\mathbf{e}}_\theta + a\ddot{\phi}\underline{\mathbf{e}}_\phi - a\dot{\theta}^2\underline{\mathbf{e}}_{r1} - a\dot{\phi}^2\underline{\mathbf{e}}_{r2}).\underline{\mathbf{e}}_\phi \;=\; -\,m_2 g(-\cos\theta\,\underline{\mathbf{e}}_{r1} + \sin\theta\,\underline{\mathbf{e}}_\theta).\underline{\mathbf{e}}_\phi \;\; - T_2\,\underline{\mathbf{e}}_{r2}.\underline{\mathbf{e}}_\phi$$

Again working out the results of the unit vector scalar products from the geometric relationships then simplifies down to;

$$m_2 a(\ddot{\theta}\cos(\phi - \theta) + \ddot{\phi} + \dot{\theta}^2\sin(\phi - \theta)) \;=\; -\,m_2 g\sin\phi$$

Note that the T_2 term has gone, since $\underline{\mathbf{e}}_{r2}.\underline{\mathbf{e}}_{\phi} = 0$ since these two vectors are perpendicular, and m_2 also cancels out. Therefore, finally;

$$\ddot{\phi} = -\ddot{\theta}\cos(\phi - \theta) - \dot{\theta}^2\sin(\phi - \theta) - (g/a)\sin\phi \qquad \textbf{A2}$$

Substituting for $\ddot{\phi}$ (phi double-dot) in equation A1 using equation A2 leads to;

$$\ddot{\theta} = -m(-\ddot{\theta}\cos(\phi - \theta) - \dot{\theta}^2\sin(\phi - \theta) - (g/a)\sin\phi)\cos(\phi - \theta) + m\dot{\phi}^2\sin(\phi - \theta) - (g/a)\sin\theta$$

On rearranging;

$$\ddot{\theta}\,[1 - m\cos^2(\phi - \theta)] = m\dot{\theta}^2\sin(\phi - \theta)\cos(\phi - \theta) + (mg/a)\sin\phi\cos(\phi - \theta) + m\dot{\phi}^2\sin(\phi - \theta) - (g/a)\sin\theta$$

And so, finally, a pair of equations suitable for numerical solutions;

$$\ddot{\theta} = m[\dot{\theta}^2\sin(\phi - \theta)\cos(\phi - \theta) + (g/a)\sin\phi\cos(\phi - \theta) + \dot{\phi}^2\sin(\phi - \theta) - (g/ma)\sin\theta] / [1 - m\cos^2(\phi - \theta)]$$

together with

$$\ddot{\phi} = -\ddot{\theta}\cos(\phi - \theta) - \dot{\theta}^2\sin(\phi - \theta) - (g/a)\sin\phi$$

Numerical Solution Using the Improved Euler Method

Note that the first of these last two equations does NOT involve $\ddot{\phi}$ (phi double-dot), and so an initial value for $\ddot{\theta}$ could be calculated only knowing initial values for θ, ϕ, $\dot{\theta}$, $\dot{\phi}$ (phi dot) as well as of course a and m. This value for $\ddot{\theta}$ can then be used in the second equation to calculate an initial value for $\ddot{\phi}$ (phi double-dot).

Thus write $\ddot{\theta} = F(\theta, \phi, \dot{\theta}, \dot{\phi})$ and write $\ddot{\phi} = G(\theta, \phi, \dot{\theta}, \dot{\phi}, \ddot{\theta})$

If θ_0, ϕ_0, $\dot{\theta}_0$, $\dot{\phi}_0$ be the initial values for these variables.

Then $\ddot{\theta}_0 = F(\theta_0, \phi_0, \dot{\theta}_0, \dot{\phi}_0)$ and $\ddot{\phi}_0 = G(\theta_0, \phi_0, \dot{\theta}_0, \dot{\phi}_0, \ddot{\theta}_0)$

And let $\Delta\dot{\theta}_{01} = \ddot{\theta}_0\Delta t$ and $\Delta\dot{\phi}_{01} = \ddot{\phi}_0\Delta t$

$\dot{\theta}_{11} = \dot{\theta}_0 + \Delta\dot{\theta}_{01}$ and $\dot{\phi}_{11} = \dot{\phi}_0 + \Delta\dot{\phi}_{01}$

$\theta_{11} = \theta_0 + (\dot{\theta}_0 + \dot{\theta}_{11})\Delta t/2$ and $\phi_{11} = \phi_0 + (\dot{\phi}_0 + \dot{\phi}_{11})\Delta t/2$

Thus, estimates for the next values of θ, ϕ, $\dot{\theta}$, and $\dot{\phi}$ have been calculated. These can now be used back in the functions $F(\theta, \phi, \dot{\theta}, \dot{\phi})$ and $G(\theta, \phi, \dot{\theta}, \dot{\phi}, \ddot{\theta})$ to calculate revised estimates for $\ddot{\theta}$ and $\ddot{\phi}$.

Specifically, $\ddot{\theta}_{11} = F(\theta_{11}, \phi_{11}, \dot{\theta}_{11}, \dot{\phi}_{11})$ and $\ddot{\phi}_{11} = G(\theta_{11}, \phi_{11}, \dot{\theta}_{11}, \dot{\phi}_{11}, \ddot{\theta}_{11})$

And so, $\Delta\dot{\theta}_{02} = \ddot{\theta}_{11}\Delta t$ and $\Delta\dot{\phi}_{02} = \ddot{\phi}_{11}\Delta t$

$\dot{\theta}_{12} = \dot{\theta}_0 + \Delta\dot{\theta}_{02}$ and $\dot{\phi}_{12} = \dot{\phi}_0 + \Delta\dot{\phi}_{02}$

$\theta_1 = \theta_0 + (\dot{\theta}_0 + \dot{\theta}_{12})\Delta t/2$ and $\phi_1 = \phi_0 + (\dot{\phi}_0 + \dot{\phi}_{12})\Delta t/2$

And $\dot{\theta}_1 = (\dot{\theta}_{11} + \dot{\theta}_{12})/2$ and $\dot{\phi}_1 = (\dot{\phi}_{11} + \dot{\phi}_{12})/2$

The values in bold are then taken as acceptable estimates for θ, ϕ, $\dot{\theta}$, $\dot{\phi}$ for the next time-period, and can be used to plot progress on the computer screen as a simulation. They are then plugged back into the above regime as input variables to calculate the *next* values, and so on.

Energy Conservation

Since the model is frictionless and ignores air resistance, the total kinetic plus potential energy of the system will be preserved in the motion. Tracking the total energy is a good way to check that the accuracy of the Improved Euler calculations is being maintained.

The total potential energy will be the sum of the potential energy of each mass. Taking the vertical resting position of the masses as the zero base, from the geometry of the diagram this amounts to;

$V = m_1ga(1 - \cos\theta) + m_2ga(2 - \cos\theta - \cos\phi)$

Similarly, the total kinetic energy will be the sum of the kinetic energy of each of the masses;

$T = \frac{1}{2}m_1\dot{\underline{\mathbf{r}}}_1^2 + \frac{1}{2}m_2\dot{\underline{\mathbf{r}}}_2^2 = \frac{1}{2}m_1(a\dot{\theta}\underline{\mathbf{e}}_\theta)^2 + \frac{1}{2}m_2(a\dot{\theta}\underline{\mathbf{e}}_\theta + a\dot{\phi}\underline{\mathbf{e}}_\phi)^2$

Therefore $\quad T = \frac{1}{2}m_1a^2\dot{\theta}^2 + \frac{1}{2}m_2a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\ \underline{\mathbf{e}}_\theta.\underline{\mathbf{e}}_\phi)$

Now, as before $\quad \underline{\mathbf{e}}_\theta.\underline{\mathbf{e}}_\phi = (\cos\theta\ \underline{\mathbf{i}} + \sin\theta\ \underline{\mathbf{j}}).(\cos\phi\ \underline{\mathbf{i}} + \sin\phi\ \underline{\mathbf{j}}) = \cos\theta\cos\phi + \sin\theta\sin\phi = \cos(\phi - \theta)$

Therefore $\quad T = \frac{1}{2}m_1a^2\dot{\theta}^2 + \frac{1}{2}m_2a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\phi - \theta))$

The total energy is then the sum $E = T + V$ so that;

$$\mathbf{E = \frac{1}{2}m_1a^2\dot{\theta}^2 + \frac{1}{2}m_2a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\phi - \theta)) + m_1ga(1 - \cos\theta) + m_2ga(2 - \cos\theta - \cos\phi)}$$

In practice, the above formulation achieves a very accurate track of energy conservation EXCEPT when the first mass, m_1, is small compared to the lower mass, m_2. The accuracy is lost when θ and ϕ approach identical values. The reason for this is the $[1 - m\cos^2(\phi - \theta)]$ denominator term in $F(\theta, \phi, \dot{\theta}, \dot{\phi})$. When $m_1 << m_2$, the factor $m = m_2/(m_1 + m_2)$ is very close to 1, as is $\cos^2(\phi - \theta)$, so that $[1 - m\cos^2(\phi - \theta)]$ is close to zero. Being a denominator in the determination of $\ddot{\theta}$ means that $\ddot{\theta}$ can become very large and uncertain. The model is then likely to start to fail. To deal with this, the time-step is shortened by a factor (1 - m), which makes the model run very slowly when m is only very slightly less than 1.