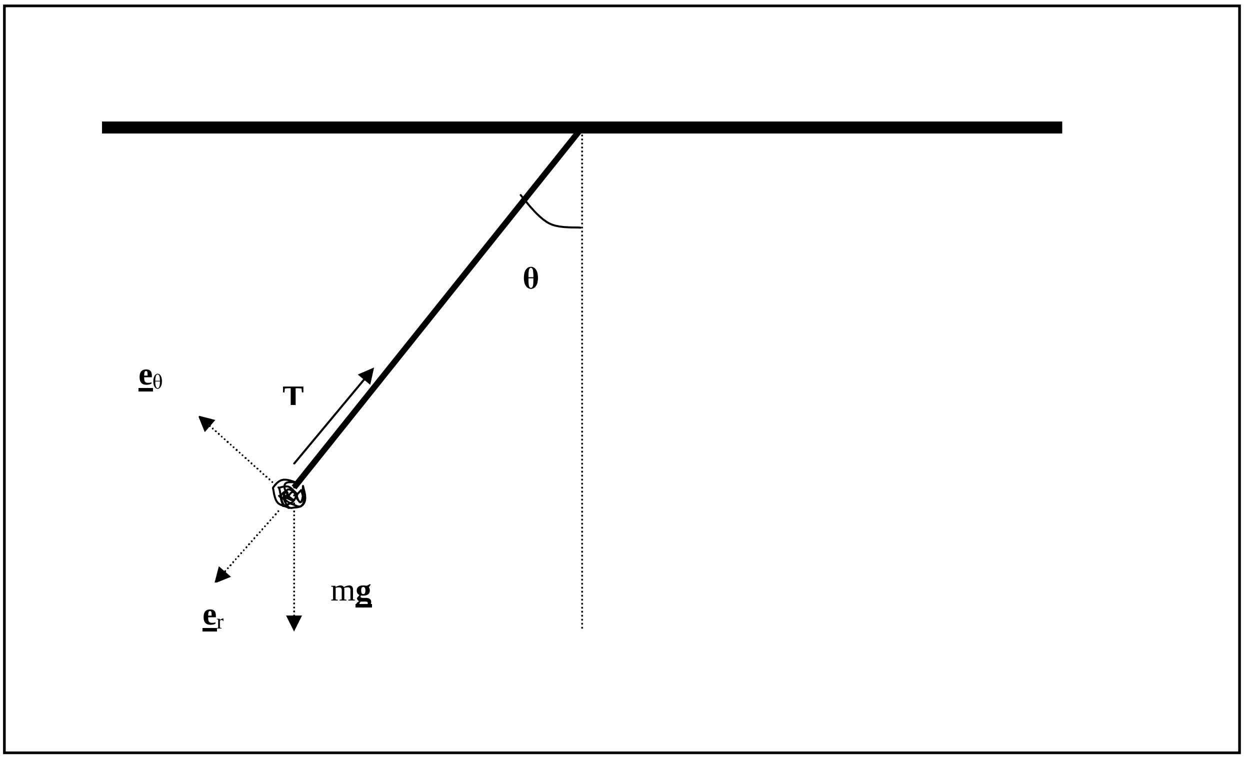
The SPRING PENDULUM – Mathematical Formulation

Formula for acceleration in Polar Co-ordinates:

$$\frac{d^2\mathbf{r}}{dt^2} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + \left(2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r\left(\frac{d^2\theta}{dt^2}\right)\right)\mathbf{e}_\theta$$

where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the unit vectors along the length of the pendulum and perpendicular to it respectively (as in the diagram below).



From the diagram above, we apply Newton's 2nd Law (Force = Mass x Acceleration) in the direction of $\mathbf{e}_r$ and in the direction of $\mathbf{e}_\theta$ as follows;

Along $\mathbf{e}_r$:

$$m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) = mg\cos\theta - T \quad \ldots\ldots\ldots\text{A}$$

Along $\mathbf{e}_\theta$:

$$m\left(2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r\left(\frac{d^2r}{dt^2}\right)\right) = -mg\sin\theta \quad \ldots\ldots\text{B}$$

Hooke's Law models the behaviour of a spring; the force is proportional to the extension beyond the "natural" unstretched length of the spring. That is,

Force, $T = \lambda (L - L_0)$ where: L_0 is the unstretched length of the spring

$= \lambda (x_0 + x)$

L is the total length of the spring at any time

x_0 is the extension caused by mass m while the spring is hanging vertically (not moving)

x is the subsequent extension of the spring during the motion (can be +ve or –ve)

λ is the Spring Modulus

Initially, if the weight is attached to the spring and allowed to hang in equilibrium, we get;

$mg = T_0 = \lambda x_0$ so that $\lambda = mg/x_0$ and $T = T_0 + \lambda x$

Now in equations A and B above, r represents the distance from the point of suspension to the bob, and so we need to replace r with $L_0 + x_0 + x$, where L_0 and x_0 are constant.

Thus $r = (L_0 + x_0 + x)$ but $\dfrac{dr}{dt} = \dfrac{dx}{dt}$ and $\dfrac{d^2r}{dt^2} = \dfrac{d^2x}{dt^2}$

Equations A and B therefore become;

C …… $\dfrac{d^2x}{dt^2} - (L_0 + x_0 + x)\left(\dfrac{d\theta}{dt}\right)^2 = g\cos\theta - (\lambda/m)(x_0 + x)$ dividing through by m, and

D …… $2\left(\dfrac{dx}{dt}\right)\left(\dfrac{d\theta}{dt}\right) + (L_0 + x_0 + x)\dfrac{d^2\theta}{dt^2} = -g\sin\theta$ where the mass, m, cancels

Now, $\dfrac{d\left\{(L_0 + x_0 + x)^2 (d\theta/dt)\right\}}{dt} = (L_0 + x_0 + x)^2 \dfrac{d^2\theta}{dt^2} + 2(L_0 + x_0 + x)\left(\dfrac{dx}{dt}\right)\left(\dfrac{d\theta}{dt}\right)$

This is just $(L_0 + x_0 + x)$ times equation D above, so that;

$$\frac{d\left\{(L_0 + x_0 + x)^2 (d\theta/dt)\right\}}{dt} = -g(L_0 + x_0 + x)\sin\theta \quad \text{……….. E}$$

Changing to Δ-notation to signify SMALL but FINITE differentials (ie real numbers used in numerical computation) equation E can be written;

$$\Delta\left\{(L_0 + x_0 + x)^2 (d\theta/dt)\right\} = -g(L_0 + x_0 + x)\sin\theta.\Delta t$$

Since we are dealing with <u>approximations</u> here, in this case ($L_0 + x_0 + x$) I'll presume that the length does not change as appreciably as the angular velocity, $d\theta/dt$, so that I'll write;

$$\Delta(d\theta / dt) = -\left(\frac{g}{(L_0 + x_0 + x)}\right)\sin\theta.\Delta t$$

so I've divided through by ($L_0 + x_0 + x$) thus assuming it to be CONSTANT for the purposes of approximate numerical calculation on the PC.

Equation C I recast as;

$$\Delta(dx / dt) = \left\{(L_0 + x_0 + x)(d\theta / dt)^2 + g\cos\theta - (\lambda / m)(x_0 + x)\right\}.\Delta t$$

These are the two equations I use to numerically solve the problem. They work fine so long as the size of x is small compared to L_0, which in reality is always the case. However in my simulation, if you choose parameters which violate this, the results are catastrophically impossible! In reality the spring would snap or buckle under extremes, but I have not modeled this eventuality.

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