The Spring Pendulum

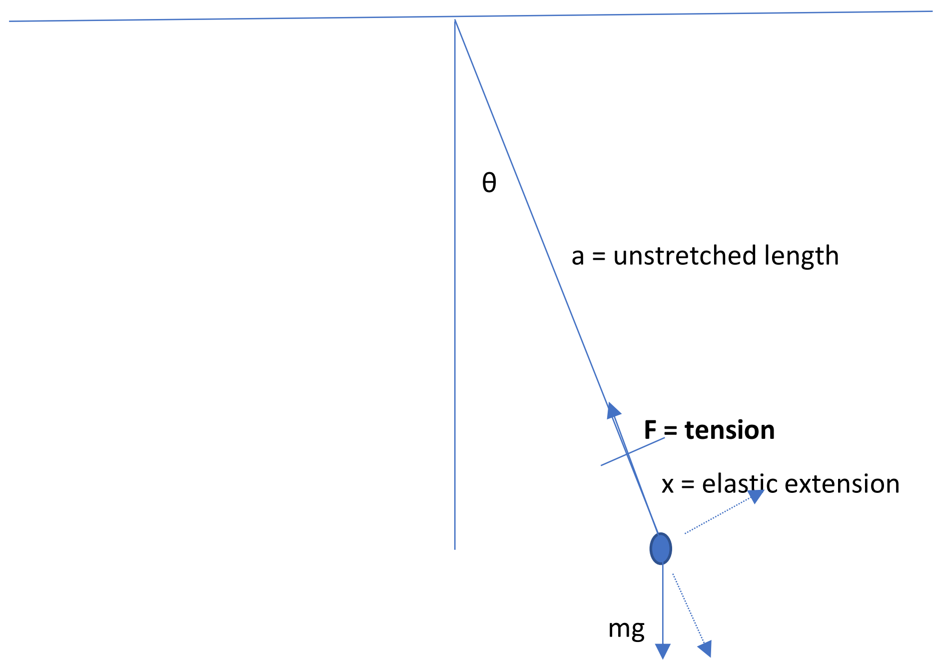
A spring pendulum is where a pendulum bob is attached to an an elastic rope or spring. I assume that the rope obeys Hooke's Law – that is, the tension in the rope is proportional to the extension so that;

$$F = \lambda x / a$$

I assume no friction or air resistance, so the system is conservative i.e. there will be no loss of total kinetic and potential energy in the motion of the pendulum.

The potential energy in a spring with force as above is;

$$PE = \lambda x^2 / a$$



From the diagram above, there are two independant and mutually perpendicular components to the velocity of the bob shown by the dotted arrows, being dx/dt along the spring and (L + x)dθ/dt perpendicular to the spring.

Therefore, the total Kinetic Energy will be;

$$T = \frac{m}{2}\left((a+x)^2 \dot{\theta}^2 + \dot{x}^2\right) \qquad \text{or} \qquad T = \frac{m}{2}\left((a+x)^2 \omega^2 + u^2\right)$$

where ω = dθ/dt and u = dx/dt

and the total Potential Energy will be;

$$V = mga - mg(a+x)\cos\theta + \lambda x^2 / 2l$$

The Lagrangian, L = T – V so that the equations of motion come from Lagrange's Equations;

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \omega}\right)-\frac{\partial L}{\partial \theta}=0 \qquad \text{and} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial u}\right)-\frac{\partial L}{\partial x}=0$$

This leads to the following 2 equations of motion (exactly the same ones as would be got from Newtonian analysis, of course)

$$\frac{d^2\theta}{dt^2}=\frac{-(2u\omega+g\sin\theta)}{(a+x)} \qquad \text{and} \qquad \frac{d^2x}{dt^2}=(a+x)\omega^2+g\cos\theta-\lambda x/2ml$$

These non-linear interlinked equations can only be solved numerically. In my program, I use the two-stage Improved Euler method. Essentially, this involves making a first-pass calculation of the next values of the speeds and position from the previous values using a small time-step of 1/1000th second. Then, a second-pass is calculated using the first-pass values. The final output is an average of the next values resulting from the first and second passes. I've not included the equations, because they're too messy. You can see them in the text version of my program, although they're not easy to follow.